

# A Poincaré Lemma for Whitney-de Rham complex

By

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## Abstract

Let  $M$  be a real analytic manifold,  $Z$  a closed subanalytic subset of  $M$ . We show that the Whitney-de Rham complex over  $Z$  is quasi-isomorphic to the constant sheaf  $\mathbb{C}_Z$ .

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Review on Whitney and formal cohomology functors</b>	<b>2</b>
<b>3</b>	<b>Main result</b>	<b>3</b>

## 1 Introduction

In [KS96], Kashiwara-Schapira introduced the Whitney functor (real case) and formal cohomology functor (complex case), then they introduced the notion of ind-sheaves and they also defined Grothendieck six operations in this framework in [KS01]. As applications, they defined the Whitney  $\mathcal{C}^\infty$  functions and Whitney holomorphic functions on the subanalytic site as examples of ind-sheaves. The more elementary study for sheaves on the subanalytic site is performed in [Pr08] and [Pr12].

Let  $M$  be a real analytic manifold, by Poincaré Lemma, it is well-known that the de Rham complex over  $M$  is isomorphic to  $\mathbb{C}_M$ . The aim of this paper is to show that a theorem of [BP08] follows easily from a deep result of Kashiwara on regular holonomic  $\mathcal{D}$ -module [K84] and the Whitney functor of [KS96]. More precisely, we show that

**Main theorem(=Theorem 3.3.)** *Let  $M$  be a real analytic manifold of dimension  $n$  and  $Z$  a closed subanalytic subset of  $M$ . Then we have:*

$$\mathbb{C}_Z \xrightarrow{\sim} (0 \longrightarrow \mathcal{W}_{M,Z}^\infty \xrightarrow{d} \mathcal{W}_{M,Z}^{(\infty,1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{W}_{M,Z}^{(\infty,n)} \longrightarrow 0)$$

where  $\mathcal{W}_{M,Z}^\infty$  denotes the sheaf of Whitney functions on  $Z$  and  $\mathcal{W}_{M,Z}^{(\infty,i)}$  denotes the sheaf of differential forms of degree  $i$  with coefficients in  $\mathcal{W}_{M,Z}^\infty$  for each  $i$ , i.e., the Whitney-de Rham complex is isomorphic to  $\mathbb{C}_Z$ .

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## 2 Review on Whitney and formal cohomology functors

In this section, we review some results on Whitney and formal cohomology functors. References are made to [KS96], [KS01], [Pr08], [Pr12] and [S12].

Let  $M$  be a real analytic manifold, we denote by  $\mathcal{A}_M, \mathcal{C}_M^\infty$  the sheaf of complex-valued real analytic functions,  $\mathcal{C}^\infty$ -functions on  $M$ . We denote by  $\mathcal{D}_M$  the sheaf of rings on  $M$  of finite-order differential operators with coefficients in  $\mathcal{A}_M$ .

We denote by  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$  the abelian category of  $\mathbb{R}$ -constructible sheaves on  $M$  and  $\text{Mod}(\mathcal{D}_M)$  the abelian category of left  $\mathcal{D}_M$ -modules. We also denote by  $\text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$  the bounded derived category consisting of objects whose cohomology groups belong to  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$  and  $\text{D}^b(\mathcal{D}_M)$  the derived category of  $\text{Mod}(\mathcal{D}_M)$  with bounded cohomologies.

**Definition 2.1.** Let  $Z$  be a closed subset of  $M$ . We denote by  $\mathcal{I}_{M,Z}^\infty$  the sheaf of  $\mathcal{C}^\infty$  functions on  $M$  vanishing up to infinite order on  $Z$ .

**Definition 2.2.** A Whitney function on a closed subset  $Z$  of  $M$  is an indexed family  $F = (F^k)_{k \in \mathbb{N}}$  consisting of continuous functions on  $Z$  such that  $\forall m \in \mathbb{N}, \forall k \in \mathbb{N}^n, |k| \leq m, \forall x \in Z, \forall \varepsilon > 0$  there exists a neighborhood such that  $\forall y, z \in U \cap Z$

$$|F^k(z) - \sum_{|j+k| \leq m} \frac{(z-y)^j}{j!} F^{j+k}(y)| \leq \varepsilon d(y, z)^{m-|k|}.$$

We denote by  $W_{M,Z}^\infty$  the space of Whitney  $\mathcal{C}^\infty$  functions on  $Z$ . We denote by  $\mathcal{W}_{M,Z}^\infty$  the sheaf  $U \mapsto W_{U, U \cap Z}^\infty$ .

In [KS96], the authors defined the Whitney tensor product functor as follows:

$$\cdot \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty : \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \rightarrow \text{Mod}(\mathcal{D}_M)$$

in the following way: let  $U$  be an open subanalytic subset of  $M$  and  $Z = M \setminus U$ . Then  $\mathbb{C}_U \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty = \mathcal{I}_{M,Z}^\infty$  and  $\mathbb{C}_Z \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty = \mathcal{W}_{M,Z}^\infty$ . This functor is exact and extends as a functor in the derived category, from  $\text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$  to  $\text{D}^b(\mathcal{D}_M)$ . Moreover the sheaf  $F \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty$  is soft for any  $\mathbb{R}$ -constructible sheaf  $F$ .

Now let  $X$  be a complex manifold and we denote by  $\mathcal{D}_X$  the sheaf of rings on  $X$  of finite-order differential operators. We still denote by  $X$  the real underlying manifold and we denote by  $\bar{X}$  the complex manifold conjugate to  $X$ . One defines the functor of formal cohomology as follows:

Let  $F \in \text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X)$ , we set

$$F \overset{w}{\otimes} \mathcal{O}_X = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{w}{\otimes} \mathcal{C}_X^\infty),$$

where  $\mathcal{D}_{\overline{X}}$  denotes the sheaf of rings on  $\overline{X}$  of finite-order differential operators.

Let  $M$  be a real analytic manifold,  $X$  a complexification of  $M$ ,  $\iota : M \hookrightarrow X$  the embedding. We recall the following result.

**Theorem 2.3.** ([KS96] Theorem 5.10.) *Let  $F \in D_{\mathbb{R}-c}^b(\mathbb{C}_M)$ . Then we have*

$$\iota_* F \overset{w}{\otimes} \mathcal{O}_X \simeq \iota_*(F \overset{w}{\otimes} \mathcal{C}_M^\infty).$$

In particular,

$$\mathbb{C}_M \overset{w}{\otimes} \mathcal{O}_X \simeq \mathcal{C}_M^\infty.$$

□

The following proposition is the key point of this note.

**Proposition 2.4.** ([KS96] Corollary 6.2) *Let  $\mathfrak{M}$  be a regular holonomic  $\mathcal{D}_X$ -module, and let  $F$  be an object of  $D_{\mathbb{R}-c}^b(\mathbb{C}_X)$ . Then, the natural morphism:*

$$(2.1) \quad R\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, F \otimes \mathcal{O}_X) \rightarrow R\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, F \overset{w}{\otimes} \mathcal{O}_X).$$

is an isomorphism.

□

### 3 Main result

Let  $X$  be a complex manifold of dimension  $n$ . We denote by  $\mathcal{D}_X$  the sheaf of rings of finite-order differential operators and  $\Theta_X$  the sheaf of vector fields on  $X$ .

First we recall the following basic result in  $\mathcal{D}_X$ -module theory.

**Proposition 3.1.** ([K03] Proposition 1.6) *The complex*

$$0 \rightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \Theta_X \rightarrow \cdots \rightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^2 \Theta_X \rightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X \rightarrow \mathcal{D}_X \rightarrow \mathcal{O}_X \rightarrow 0$$

is exact.

□

**Lemma 3.2.** *Let  $\mathcal{M}$  be a left  $\mathcal{D}_X$ -module. Then we have*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}) \simeq [\mathcal{M} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow \cdots \rightarrow \bigwedge^n \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M}].$$

*Proof.* By Proposition 3.1, we have

$$\begin{aligned} & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}) \\ & \simeq [\mathcal{M} \rightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X, \mathcal{M}) \rightarrow \cdots \rightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \Theta_X, \mathcal{M})] \\ & \simeq [\mathcal{M} \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\Theta_X, \mathcal{M}) \rightarrow \cdots \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\bigwedge^n \Theta_X, \mathcal{M})] \\ & \simeq [\mathcal{M} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow \cdots \rightarrow \bigwedge^n \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M}] \end{aligned}$$

where  $\Omega_X^1 := \mathcal{H}om_{\mathcal{O}_X}(\Theta_X, \mathcal{O}_X)$ .  $\square$

Let  $M$  be a real analytic manifold,  $X$  a complexification of  $M$  and  $Z$  a closed subanalytic subset of  $M$ . We denote by  $\Omega_X^1$  the sheaf of differential one-form on  $X$  and

$$(3.1) \quad \mathcal{A}_Z^{(i)} := \bigwedge^i \Omega_X^1 \otimes_{\mathcal{O}_X} (\mathbb{C}_Z \otimes \mathcal{O}_X),$$

$$(3.2) \quad \mathcal{W}_{M,Z}^{(\infty,i)} := \bigwedge^i \Omega_X^1 \otimes_{\mathcal{O}_X} (\mathbb{C}_Z \overset{\text{w}}{\otimes} \mathcal{O}_X) \simeq \bigwedge^i \Omega_X^1 \otimes_{\mathcal{O}_X} (\mathbb{C}_Z \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty).$$

Now we are ready to prove the main theorem of this note below.

**Theorem 3.3.** *Let  $M$  be a real analytic manifold of dimension  $n$  and  $Z$  a closed subanalytic subset of  $M$ . Then we have:*

$$\mathbb{C}_Z \xrightarrow{\sim} (0 \longrightarrow \mathcal{W}_{M,Z}^\infty \xrightarrow{d} \mathcal{W}_{M,Z}^{(\infty,1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{W}_{M,Z}^{(\infty,n)} \longrightarrow 0)$$

where  $\mathcal{W}_{M,Z}^\infty$  denotes the sheaf of Whitney functions on  $Z$  and  $\mathcal{W}_{M,Z}^{(\infty,i)}$  denotes the sheaf of differential forms of degree  $i$  with coefficients in  $\mathcal{W}_{M,Z}^\infty$  for each  $i$  which are defined in (3.2), i.e., the Whitney-de Rham complex is isomorphic to  $\mathbb{C}_Z$ .

*Proof.* Take  $\mathfrak{M} = \mathcal{O}_X$  and  $F = \mathbb{C}_Z$  in Proposition 2.4.

On the one hand, we show that the left hand side of (2.1) is  $\mathbb{C}_Z$ . By Theorem 2.3 and Lemma 3.2, we get the following complex

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \mathcal{A}_M^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_M^{(n)} \longrightarrow 0$$

which is exact by Poincaré lemma where  $\mathcal{A}_M^{(i)}$ 's are defined in (3.1) by taking  $Z = M$ . Tensoring  $\mathbb{C}_Z$ , we obtain the following exact sequence

$$0 \longrightarrow \mathbb{C}_Z \longrightarrow \mathcal{A}_Z^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_Z^{(n)} \longrightarrow 0.$$

Therefore,

$$\mathbb{C}_Z \xrightarrow{\sim} (0 \longrightarrow \mathcal{A}_Z^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_Z^{(n)} \longrightarrow 0).$$

On the other hand, the right hand side of (2.1) is the Whitney-de Rham complex

$$0 \longrightarrow \mathcal{W}_{M,Z}^\infty \xrightarrow{d} \mathcal{W}_{M,Z}^{(\infty,1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{W}_{M,Z}^{(\infty,n)} \longrightarrow 0.$$

Now the result follows from the isomorphism of (2.1).  $\square$

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